211. (April, 1914.) Proposed by E. T. BELL, University of Washington.

If an odd perfect number exists, the total number of its divisors is a multiple of 2, but not of 4; or, what is the same thing, an odd perfect number must be of the form $p^{2a-1}n^2$, where p is prime and a is odd.

214. (April, 1914.) Proposed by A. J. KEMPNER, University of Illinois.

Let a be a positive integer $\geqq 2$, and let $T(n)$ denote the number of distinct divisors of the positive integer n, including both 1 and n, so that $T(1) = 1$, $T(2) = 2$, $T(3) = 2$, $T(4) = 3$, $\cdots$. Show that

$$\Sigma_{n=1}^{n=\infty} T(n)/a^n = \Sigma_{n=1}^{n=\infty} 1/(a^n - 1).$$

The special case $a = 10$ gives, as is easily seen:

$$9\sum_{n=1}^{n=\infty} \frac{T(n)}{10^n} = \frac{1}{1} + \frac{1}{11} + \frac{1}{111} + \frac{1}{1111} + \cdots.$$

217. (May, 1914.) Proposed by E. T. BELL, University of Washington.

(i) If r is a prime greater than 2, and $p \equiv 2^a r + 1$ is prime, the only solution, when n is greater than 2, of $x^n - y^n = p$, is $n = 3$, $x = 2$, $y = 1$.

(ii) The only primes that are simultaneously of the forms $4k + 1$ and $3^m - 2^m$ are 1 and 5.

(iii) Generalize (ii).

219. (June, 1914.) Proposed by R. D. CARMICHAEL, University of Illinois.

Determine whether it is possible for a polygon to have the number of its diagonals equal to a perfect fourth power.

221. (September, 1914.) Proposed by T. E. MASON, Purdue University.

Find a number x such that the sum of the divisors of x is a perfect square. [Carmichael, *Theory of Numbers*, page 17.]

222. (October, 1914.) Proposed by A. H. HOLMES, Brunswick, Me.

Find rational values for m and n such that $(m^2 + 1)/m^2 + (n^2 + 1)/n^2$ may be the square of an integer.

223. (October, 1914.) Proposed by T. E. MASON, Purdue University.

Show that

$$\frac{(rst)!}{t!(s!)^t(r!)^{st}}$$

is an integer, r, s, and t being positive integers. Generalize to the case of n integers, $r, s, t, u, \cdots$. [Carmichael, *Theory of Numbers*, page 28.]

SOLUTIONS OF PROBLEMS.

ALGEBRA.

443. Proposed by A. M. KENYON, Purdue University.

If p_r denote the sum of all the r-factor products that can be formed from the first n natural numbers ($p_r = 0$ for $r > n$), and if

$$D_s = \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 \\ 2p_2 & p_1 & 1 & \cdots & 0 \\ 3p_3 & p_2 & p_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ sp_s & p_{s-1} & p_{s-2} & \cdots & p_1 \end{vmatrix}$$

show that

$$\sum_{i=0}^{k} (-1)^i c_i \binom{k}{i} D_{2k-i} = 0, \qquad k, n = 1, 2, 3, \cdots,$$

where

$$c_i = \frac{2k+1-i}{1+i}$$

when i is even and $(2n+1)$ when i is odd; and $\binom{k}{i}$ is the coefficient of x^i in $(1+x)^k$.

Solution by the Proposer.

The roots of the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \cdots + (-1)^n p_n = 0$$

are the natural numbers 1, 2, 3, $\cdots n$.

Solving Newton's formulæ[1] for the sums of like powers of the roots, we obtain

$$1^s + 2^s + 3^s + \cdots + n^s = D_s, \qquad n, s = 1, 2, 3, \cdots,$$

A relation between the D's of odd subscript has been published[2] which is equivalent to

$$\text{(1)} \qquad \sum_{i=0}^{I(k/2)} \binom{k}{2i+1} D_{2k-1-2i} = 2^{k-1} D_1^k, \qquad k, n = 1, 2, 3, \cdots,$$

and the following relation[3] exists among the D's of even subscript,

$$\text{(2)} \qquad \sum_{i=0}^{I(k/2)} \frac{2k+1-2i}{1+2i} \binom{k}{2i} D_{2k-2i} = (2n+1)2^{k-1} D_1^k, \qquad k, n = 1, 2, 3, \cdots.$$

These formulæ, in which $I(k/2)$ denotes the integral part of $k/2$, are readily established by induction. Multiplying (1) by $2n+1$ and subtracting the result from (2) we get the formula sought.

444. Proposed by J. E. ROWE, Pennsylvania State College.

Prove that the determinant

$$\begin{vmatrix} \cot A, & \cot B, & \cot C \\ 1, & 1, & 1 \\ \cos^2 A, & \cos^2 B, & \cos^2 C \end{vmatrix} = 0, \qquad \text{if} \qquad A + B + C = 180°.$$

Solution by S. E. Rasor, Ohio State University.

Transforming trigonometrically and rearranging, the determinant becomes

$$\frac{-1}{4 \sin A \sin B \sin C} \begin{vmatrix} 2\cos A \sin B \sin C, & 2 \cos B \sin C \sin A, & 2 \cos C \sin A \sin B \\ 1, & 1, & 1 \\ 2\sin^2 A, & 2 \sin^2 B, & 2\sin^2 C \end{vmatrix}.$$

By the formula $2 \sin A \sin B \cos C = \sin^2 A + \sin^2 B - \sin^2 C$ when $A + B + C = 180°$, this reduces to

$$\frac{-1}{4 \sin A \sin B \sin C} \begin{vmatrix} \sin^2 B + \sin^2 C - \sin^2 A, & \sin^2 A + \sin^2 C - \sin^2 B, & \sin^2 A + \sin^2 B - \sin^2 C \\ 1, & 1, & 1 \\ 2\sin^2 A, & 2 \sin^2 B, & 2\sin^2 C \end{vmatrix},$$

[1] See, for example, Cajori's *Theory of Equations*, pages 85–86.

[2] Stern, *Crelle's Journal*, volume 84, pages 216–218.

[3] *Proceedings Indiana Academy of Sciences*, 1914, page 440.